Moving horizon ℓ_2 control of LPV systems subject to constraints ⋆

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Abstract: The paper presents a moving horizon ℓ_2 control approach for the class of linear parameter-varying systems. By solving online a convex optimization problem subject to linear matrix inequality constraints the ℓ_2 gain from the energy bounded external disturbance to the performance output is minimized at each sampling instant. The approach guarantees satisfaction of state and input constraints and it is shown that the online recalculation of the control law improves disturbance attenuation significantly when compared to a static control law.

1. INTRODUCTION

Model predictive control (MPC), receding horizon control (RHC), or moving horizon control (MHC) is an optimization based control method. In its general form a control sequence is determined by optimizing a finite horizon cost function at each sampling instant, based on an explicit model of the considered system and the current state measurement. The first part of the obtained control input is applied to the system. At the next sampling instant, the optimization problem is solved again based on new state measurements, and the control input is updated. Due to its ability to explicitly handle state and input constraints, MHC has received much interest in both academic community and industrial applications over the last decades, see e.g. Mayne et al. [2000], Qin and Badgwell [2003].

In many control problems one has to deal with linear parameter-varying (LPV) systems. They represent a class of nonlinear systems which can be controlled using linear-like control techniques. The dynamics of LPV systems depend on a time-varying parameter, which takes values in pre-specified sets. The analysis and synthesis of LPV systems play an important role in control theory and application since they can be applied to both nonlinear systems and linear systems with model uncertainties. Many research activities have focussed on the development of control methods for LPV systems in the past, see for example the results presented by Shamma and Athans [1991], Apkarian et al. [1995], Scherer and Weiland [2000], Wu [2001], Blanchini and Miani [2003], Bliman [2005] and Lim [1999] for an overview. Since moving horizon control has well-known advantageous properties such as optimal solutions with respect to the considered cost function and guaranteed satisfaction of state and input constraints, see e.g. Mayne et al. [2000], clearly also several MHC schemes that are able to deal with LPV systems have been published in the literature, see e.g. Casavola et al. [2002], Cuzzola et al. [2002], Kothare et al. [1996], Lee et al. [2007], Lu and Arkun [2000], Park and Jeong [2004], Pluymers et al. [2005] and Yu et al. [2009a]. In most of those methods the control law is calculated by repeatedly solving a convex optimization problem based on linear matrix inequalities (LMIs, Boyd et al. [1994]) such that an upper bound on a worst-case cost function is minimized. Most of these approaches consider LPV systems which might be uncertain, however are not affected by external disturbances. In the presence of energy bounded external disturbances the concept of ℓ_2 stability (Khalil [2002]) is a suitable method to tackle the problem. To the authors' best knowledge there exist only few results on moving horizon control with guaranteed ℓ_2 performance for LPV systems (Lee and Park [2008]). The goal of this paper is to extend the results of Chen and Scherer [2006] and Yu et al. [2009a] to design a novel MHC controller with guaranteed ℓ_2 performance for state and input constrained discrete-time LPV systems. Similar to Yu et al. [2009a] a parameter-dependent state feedback control law is applied which is based on the repeated solution of a convex optimization problem subject to LMI constraints. The LMI conditions are derived from a dissipation inequality as it is used in Chen and Scherer [2006] and Khalil [2002]. By solving the optimization problem at each sampling instant the ℓ_2 gain from the disturbance to the considered performance output is minimized.

The remainder of the paper is structured as follows. In Section 2 the considered system class is introduced and the control task is discussed. Section 3 provides the main result of the paper, namely a moving horizon ℓ_2 controller for LPV systems with guaranteed satisfaction of state and input constraints. A simulation example in Section 4 illustrates the effectiveness of the moving horizon approach. The paper is concluded in Section 5 with a brief summary.

1.1 Notation

We denote $\psi_{i,k}$ as the i-th element of the vector ψ_k. With I and 0 we denote an identity matrix and a zero matrix, respectively, of suitable dimension. The vectors

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e_m, $m = 1, \ldots, m_{max}$, represent the column vectors of an identity matrix of dimension $m_{max} \times m_{max}$. With the expression $Co\{F_1, \ldots, F_N\}$ we denote the convex hull of the N matrices $F_1, \ldots, F_N$.

Remark 1. In this paper we use the expression ℓ_2, and not as in many other publications the expression $\mathcal{H}_\infty$. In the linear case the ℓ_2 gain of a system is equivalent to its $\mathcal{H}_\infty$ gain. However, the $\mathcal{H}_\infty$ norm is defined in the frequency domain. Since the frequency domain generally is only defined for linear systems, one would have to clearly define the $\mathcal{H}_\infty$ norm for nonlinear systems, and in this paper in particular for LPV systems. Furthermore, we provide time-domain considerations in this paper. Therefore, it is more suitable to use the ℓ_2 expression instead of $\mathcal{H}_\infty$.

2. PROBLEM SETUP

We consider discrete-time linear parameter-varying (LPV) systems of the form

$$x_{k+1} = A(\theta_k)x_k + B(\theta_k)u_k + G(\theta_k)w_k, \tag{1a}$$

$$y_k = C(\theta_k)x_k + D(\theta_k)u_k + H(\theta_k)w_k, \tag{1b}$$

$$z_k = E(\theta_k)x_k + F(\theta_k)u_k, \tag{1c}$$

subject to the constraints

$$-z_{m,max} \leq z_{m,k} \leq z_{m,max}, \; m = 1, \cdots, n_z, \tag{2}$$

where $x_k \in \mathbb{R}^{n_x}$ denotes the system states, $u_k \in \mathbb{R}^{n_u}$ is the control input, $w_k \in \mathbb{R}^{n_w}$ is the disturbance input satisfying

$$\sum_{k=0}^{\infty} \|w_k\|^2 \leq \beta, \tag{3}$$

i.e. the signal w_k is in $\ell_2[0, \infty)$. $y_k \in \mathbb{R}^{n_y}$ represents the performance output and $z_k \in \mathbb{R}^{n_z}$ is the constraints output. The constant vector z_{max} defines the constraints on the states x_k and the control input u_k for system (1). The system matrices $A(\theta_k) \in \mathbb{R}^{n_x \times n_x}$, $B(\theta_k) \in \mathbb{R}^{n_x \times n_u}$, $C(\theta_k) \in \mathbb{R}^{n_y \times n_x}$, $D(\theta_k) \in \mathbb{R}^{n_y \times n_u}$, $E(\theta_k) \in \mathbb{R}^{n_z \times n_x}$, $F(\theta_k) \in \mathbb{R}^{n_z \times n_u}$, $G(\theta_k) \in \mathbb{R}^{n_x \times n_w}$, and $H(\theta_k) \in \mathbb{R}^{n_y \times n_w}$ are assumed to depend on the parameter vector $\theta_k := [\theta_{1,k}, \theta_{2,k}, \cdots, \theta_{N,k}]^T \in \mathbb{R}^N$, which belongs to the polytope $\mathcal{P}$ defined by

$$\sum_{j=1}^{N} \theta_{j,k} = 1, 0 \leq \theta_{j,k} \leq 1. \tag{4}$$

We assume that the parameter θ_k can be measured online. Clearly, as θ_k varies inside the polytope $\mathcal{P}$, the matrices of system (1) vary inside the polytope Ω

$$\begin{bmatrix} A(\theta_k) & B(\theta_k) & G(\theta_k) \\ C(\theta_k) & D(\theta_k) & H(\theta_k) \\ E(\theta_k) & F(\theta_k) & 0 \end{bmatrix} \in \Omega, \tag{5}$$

which is defined by the convex hull of N local extremal matrices $A_i, B_i, C_i, D_i, E_i, F_i, G_i, H_i$, $i = 1, \cdots, N$:

$$\Omega := Co\left\{ \begin{bmatrix} A_1 & B_1 & G_1 \\ C_1 & D_1 & H_1 \\ E_1 & F_1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} A_N & B_N & G_N \\ C_N & D_N & H_N \\ E_N & F_N & 0 \end{bmatrix} \right\}.$$

Therefore, we can write the matrices of system (1) as

$$A(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} A_j, \; B(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} B_j \tag{6a}$$

$$C(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} C_j, \; D(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} D_j \tag{6b}$$

$$E(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} E_j, \; F(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} F_j \tag{6c}$$

$$G(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} G_j, \; H(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} H_j. \tag{6d}$$

The control task is to design a controller for system (1) such that the ℓ_2 gain from the disturbance w_k to the performance output y_k is minimized while the constraint output z_k satisfies (2) for all k.

In the following section we derive LMI conditions to calculate a parameter-dependent feedback law

$$u_k = K(\theta_k)x_k, \tag{7}$$

which satisfies the given control objectives, in a moving horizon manner, i.e. the control law is recalculated at each sampling instant k by the online solution of a convex optimization problem subject to the derived LMI conditions.

3. MOVING HORIZON ℓ_2 CONTROL

Suppose that $K_j \in \mathbb{R}^{n_u \times n_x}$ is a time-invariant feedback gain of the j-th vertex system. A suitable parameter-dependent feedback law for the whole LPV system (1) is obtained via the weighted average of the control laws designed for each vertex

$$K(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} K_j. \tag{8}$$

Using control law (7), for system (1) we obtain the closed-loop representation

$$x_{k+1} = A_{cl}(\theta_k)x_k + G(\theta_k)w_k, \tag{9a}$$

$$y_k = C_{cl}(\theta_k)x_k + H(\theta_k)w_k, \tag{9b}$$

$$z_k = E_{cl}(\theta_k)x_k, \tag{9c}$$

where the matrices $A_{cl}(\theta_k)$, $C_{cl}(\theta_k)$, and $E_{cl}(\theta_k)$ are

$$A_{cl}(\theta_k) = \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{i,k}\theta_{j,k}(A_i + B_i K_j), \tag{10a}$$

$$C_{cl}(\theta_k) = \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{i,k}\theta_{j,k}(C_i + D_i K_j), \tag{10b}$$

$$E_{cl}(\theta_k) = \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{i,k}\theta_{j,k}(E_i + F_i K_j). \tag{10c}$$

Based on the closed-loop system representation (9) with the closed-loop matrices defined in (10), in the following subsection 3.1 we derive some basic ℓ_2 stability results for the considered LPV system. In subsection 3.2 we exploit these results in the design of the moving horizon ℓ_2 controller.

3.1 Constrained ℓ_2 Control

The following theorem derives conditions for the calculation of a controller for system (1) such that the ℓ_2 gain from the disturbance w_k to the performance output y_k is minimized, and the output z_k satisfies the constraints (2).

Theorem 1. For a given α, suppose that there exist a symmetric, positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$, matrices $Y_1, \ldots, Y_N \in \mathbb{R}^{n_u \times n_x}$ and a constant $\gamma \in \mathbb{R}^+$ such that the optimization problem

$$\underset{\gamma, X, Y_1, \ldots, Y_N}{\text{minimize}} \ \gamma \tag{11a}$$

subject to

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{i,k}\theta_{j,k} L_{ij} \geq 0 \tag{11b}$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{i,k}\theta_{j,k} F_{ij,m} \geq 0, \ m = 1, \ldots, n_z, \tag{11c}$$

has a feasible solution for all $\theta_k \in \mathcal{P}$. Then with $K_j = Y_j X^{-1}$, $P = X^{-1}$, $K(\theta_k) = \sum_{j=1}^{N} \theta_{i,k} K_j$, $V(x) = x^T P x$ and with the matrices

$$L_{ij} = \begin{bmatrix} X & \star & \star & \star \\ 0 & \gamma I & \star & \star \\ A_i X + B_i Y_j & G_i & X & \star \\ C_i X + D_i Y_j & H_i & 0 & \gamma I \end{bmatrix}, \tag{12}$$

$$F_{ijm} = \begin{bmatrix} X & \star \\ e_m^T (E_i X + F_i Y_j) & \frac{1}{\alpha} z_{m,max}^2 \end{bmatrix}, \tag{13}$$

the state feedback law $u_k = K(\theta_k) x_k$ guarantees that

(1) the ℓ_2 gain from the disturbance w_k to the performance output y_k is less than (or equal to) γ,

(2) if the initial state x_0 satisfies $\gamma\beta + V(x_0) \leq \alpha$, then
  (a) all perturbed state trajectories remain in an ellipsoid defined as
  $$\Omega(P, \alpha) := \{x \in \mathbb{R}^{n_x} | V(x) \leq \alpha\}, \tag{14}$$
  (b) the constraints (2) are satisfied.

Proof. *Part I. Consider the Lyapunov function candidate $V(x_k) = x_k^T P x_k > 0$. We have to show that the ℓ_2 gain of the closed-loop system (9) from the disturbance w_k to the performance output y_k is bounded by γ, which is according to Lin and Byrnes [1994, 1996] the case if the dissipation inequality*

$$V(x_{k+1}) - V(x_k) + \gamma^{-1} y_k^T y_k - \gamma w_k^T w_k \leq 0 \tag{15}$$

holds for all $k \geq 0$. Substituting X and Y_j in (11b) as defined in the theorem by P and K_j, and using (6d) and (10) it follows that

$$\begin{bmatrix} P^{-1} & \star & \star & \star \\ 0 & \gamma I & \star & \star \\ A_{cl}(\theta_k) P^{-1} & G(\theta_k) & P^{-1} & \star \\ C_{cl}(\theta_k) P^{-1} & H(\theta_k) & 0 & \gamma I \end{bmatrix} \geq 0 \tag{16}$$

is satisfied. Pre- and post-multiplying with diag(P, I, P, I), applying the Schur complement, and using the system dynamics (9), leads to (15).

Part II-a. By the assumption (3), i.e. $\sum_{k=0}^{\infty} \|w_k\|^2 \leq \beta$, the dissipation inequality (15) implies that

$$V(x_k) + \gamma^{-1} \sum_{i=0}^{k-1} \|y_k\|^2 \leq V(x_0) + \gamma\beta \leq \alpha, \tag{17}$$

holds for all $k \geq 0$. Given any x_0, (17) shows that the state trajectory starting from x_0 stays in the ellipsoid $\Omega(P, \alpha)$. This implies that this ellipsoid contains the set of all reachable states for the closed-loop system starting from x_0.

Part II-b. From (2) it follows that the inequality

$$\frac{x_k^T E_{cl}^T(\theta_k) e_m e_m^T E_{cl}(\theta_k) x_k}{z_{m,max}^2} \leq 1, \ m = 1, \ldots, n_z, \tag{18}$$

has to be satisfied. From part II-a we know that $x_k \in \Omega(P, \alpha)$ for all $k \geq 0$, i.e. $x_k^T P \alpha^{-1} x_k \leq 1$ holds. Thus, satisfaction of

$$\frac{E_{cl}^T(\theta_k) e_m e_m^T E_{cl}(\theta_k)}{z_{m,max}^2} \leq P\alpha^{-1}, \ m = 1, \ldots, n_z, \tag{19}$$

implies satisfaction of the constraints (2). Substituting X and Y_j in (11c) as defined in the theorem by P and K_j, and using (10c) leads to

$$\begin{bmatrix} P^{-1} & \star \\ e_m^T E(\theta_k) P^{-1} & \frac{1}{\alpha} z_{m,max}^2 \end{bmatrix} \geq 0, \ m = 1, \ldots, n_z. \tag{20}$$

Multiplying (20) with diag(P, I) and applying the Schur complement we obtain the desired inequality (19).

In (11) the constant α does not represent an optimization variable and has to be chosen suitably before solving the optimization problem. Cleary, from (11c) we know that the choice of a larger α leads to a smaller set of feasible solutions $(Q, Y_1, \ldots, Y_N)$, and hence to introduction of conservativeness and a larger optimal value γ. This implies worse control performance. However, on the other hand it follows from the condition $\gamma\beta + V(x_0) \leq \alpha$ in part II of Theorem 1 that the smaller the choice of α, the smaller the disturbance energy β allowed for guaranteeing satisfaction of the constraints (2). This contradiction motivates us to apply a moving horizon strategy to online deal with the trade-off between constraint satisfaction and good performance. Therefore, we extend and reformulate optimzation problem (11) such that a convex optimization problem subject to LMIs is yielded, which enables a fast online solution using numerical solvers such as e.g. Sedumi (Sturm [1999]) and Sdpt3 (Tütüncü et al. [2003]).

The optimization problem (11) is subject to the parameter-dependent matrix inequality constraints (11b) and (11c). Since the desired optimal solution has to hold for all $\theta_k \in \mathcal{P}$, it is not possible to solve the given problem. The following lemma gives conditions to reformulate the optimization problem (11) in terms of parameter-independent LMIs.

Lemma 1. (Gao [2006], Kim and Lee [2000])
If there exist matrices $\Lambda_{ij} = \Lambda_{ji}^T$, $i = 1, \ldots, N$, $j = 1, \ldots, N$, such that the LMIs

$$\Gamma_{ii} \geq \Lambda_{ii}, \quad i = 1, \ldots, N, \tag{21a}$$

$$\Gamma_{ij} + \Gamma_{ji} \geq \Lambda_{ij} + \Lambda_{ij}^T, \quad i = 1, \ldots, N, \ j < i, \tag{21b}$$

$$[\Lambda_{ij}]_{N \times N} \geq 0, \tag{21c}$$

are satisfied, where

$$[\Lambda_{ij}]_{N \times N} = \begin{bmatrix} \Lambda_{11} & \cdots & \Lambda_{1N} \\ \vdots & \ddots & \vdots \\ \Lambda_{N1} & \cdots & \Lambda_{NN} \end{bmatrix}, \tag{22}$$

then with $\xi_{i,k} \geq 0$, $\sum_{i=1}^{N} \xi_{i,k} = 1\ \forall k$, *the parameter-dependent matrix inequalities*

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_{i,k}\xi_{j,k}\Gamma_{ij} \geq 0, \tag{23}$$

are satisfied for all k.

Lemma 1 allows to formulate LMI conditions as in (21) such that a parameter-dependent matrix inequality of the form (23) is satisfied. This can be used to guarantee satisfaction of the parameter-dependent matrix inequalities (11b) and (11c) by the satisfaction of LMI conditions, yielding a convex and parameter-independent optimization problem which can be solved efficiently online. Similarly, Lemma 1 has been applied in Yu et al. [2009a] and Yu et al. [2009b].

3.2 Moving Horizon Formulation

In this subsection we introduce a moving horizon ℓ_2 controller which relies on the following convex optimization problem that is solved repeatedly at each sampling instant k for given α and β:

$$\underset{\gamma_k, X_k, Y_{j,k}, T_{ij,k}, M_{ijm,k}}{\text{minimize}} \gamma_k \tag{24a}$$

subject to

$$\begin{bmatrix} \alpha - \gamma_k\beta & x_k^T \\ x_k & X_k \end{bmatrix} \geq 0, \tag{24b}$$

$$\begin{bmatrix} p_0 - p_{k-1} + x_k^T P_{k-1} x_k & x_k^T \\ x_k & X_k \end{bmatrix} \geq 0, \tag{24c}$$

$$L_{ii,k} \geq T_{ii,k},\ i = 1, \ldots, N, \tag{24d}$$

$$L_{ij,k} + L_{ji,k} \geq T_{ij,k} + T_{ij,k}^T,\ j < i = 1, \ldots, N, \tag{24e}$$

$$[T_{ij,k}]_{N\times N} \geq 0, \tag{24f}$$

$$F_{iim,k} \geq M_{iim,k},\ i = 1, \ldots, N,\ m = 1, \ldots, n_z, \tag{24g}$$

$$F_{ijm,k} + F_{jim,k} \geq M_{ijm,k} + M_{ijm,k}^T,$$
$$j < i = 1, \ldots, N,\ , m = 1, \ldots, n_z, \tag{24h}$$

$$[M_{ijm,k}]_{N\times N} \geq 0. \tag{24i}$$

The index k in the optimization variables ($\gamma_k, X_k, Y_{jk}, T_{ij,k}, M_{ijm,k}$) denotes the association with the corresponding time instant when the optimization problem is solved. The matrices $L_{ij,k}$ and $F_{ijm,k}$ are as defined in (12) and (13). Thus, using Lemma 1 we know that satisfaction of the LMIs (24d)-(24i) implies satisfaction of the matrix inequalities (11b) and (11c). Therefore, the solution to the optimization problem (24) guarantees that the properties of Theorem 1 hold. As in Theorem 1 we define $P_k = X_k^{-1}$ and $K_{j,k} = Y_{j,k}X_k^{-1}$. Thus, applying the Schur complement to (24b) leads to

$$x_k^T P_k x_k + \gamma_k\beta \leq \alpha, \tag{25}$$

i.e. the state x_k lies in the ellipsoid $\Omega(P_k, \alpha - \gamma_k\beta)$. The scalar p_k in (24c) is recursively updated as

$$p_k := p_{k-1} - x_k^T P_{k-1} x_k + x_k^T P_k x_k. \tag{26}$$

Note that according to the moving horizon principle the optimization problem (24) depends on the current system state x_k. The proposed controller is given by the following algorithm.

Algortihm 1. *The moving horizon* ℓ_2 *controller for system (1) is as follows:*

- *Step 0: Choose, respectively determine, the parameters* α *and* β *for the LMI (24b) of the optimization problem (24).*
- *Step 1: At time* $k = 0$*, get* x_0*, solve (24) without (24c) to obtain* $(\gamma_0, P_0, K_{j,0})$*. Compute* $p_0 = x_0^T P_0 x_0$ *and go to step 3.*
- *Step 2: At time* $k > 0$*, get* x_k *and solve the optimization problem (24).*
- *Step 3: Compute* p_k *according to (26). Measure the parameter vector* θ_k*. Apply the closed-loop control law*

$$u_k = \sum_{i=1}^{N} \theta_{j,k} K_{j,k} x_k \tag{27}$$

to system (1), replace k *by* $k + 1$ *and go to step 2.*

Theorem 2. For given α and β, suppose that there exists an optimal solution $(\gamma_k, X_k, Y_{j,k}, T_{ij,k}, M_{ijm,k})$ to the convex optimization problem (24) depending on the current system state x_k at each time instant k. Then the parameter-dependent control law (27) guarantees that

(1) the dissipation inequality

$$\sum_{i=0}^{k}\left(\bar{\gamma}^{-1} y_i^T y_i - \bar{\gamma} w_i^T w_i\right) \leq x_0^T P_0 x_0 \tag{28}$$

is satisfied with $\bar{\gamma} := \max\{\gamma_0, \gamma_1, \ldots, \gamma_k\}$,

(2) the ℓ_2 gain from the disturbance w_k to the performance output y_k is less than $\bar{\gamma}$,

(3) the constraints (2) are satisfied.

Proof. *Part I. From the application of Lemma 1 to the LMIs (24d)-(24f) we know from Theorem 1 that the dissipation inequality (15) is satisfied at each sampling instant* k*. Summing up (15) from* $i = 0$ *to* $i = k$ *we obtain*

$$\sum_{i=0}^{k} x_i^T P_i x_i - \sum_{i=1}^{k} x_i P_{i-1} x_i - x_{k+1}^T P_k x_{k+1}$$
$$+ \sum_{i=0}^{k}(\gamma_i w_i^T w_i - \gamma_i^{-1} y_i^T y_i) \geq 0. \tag{29}$$

From the recursive definition of p_k *in (26) it follows*

$$p_k = \sum_{i=0}^{k} x_i^T P_i x_i - \sum_{i=1}^{k} x_i^T P_{i-1} x_i. \tag{30}$$

Thus, (29) is equivalent to

$$p_k - x_{k+1}^T P_k x_{k+1} + \sum_{i=0}^{k}(\gamma_i w_i^T w_i - \gamma_i^{-1} y_i^T y_i) \geq 0. \tag{31}$$

By applying the Schur complement to the LMI (24c) and using the definition of p_k *(26) we obtain*

$$p_0 = x_0^T P_0 x_0 \geq p_k. \tag{32}$$

Therefore, the inequality

$$x_0^T P_0 x_0 \geq \sum_{i=0}^{k}(\gamma_i^{-1} y_i^T y_i - \gamma_i w_i^T w_i) + x_{k+1}^T P_k x_{k+1} \tag{33}$$

follows from (31). Since $\bar{\gamma} := \max\{\gamma_0, \gamma_1, \ldots, \gamma_k\}$*, we know that* $\bar{\gamma} \geq \gamma_i$ *and* $\bar{\gamma}^{-1} \leq \gamma_i^{-1}$ *for all* i*. Hence, with* $P_k > 0$ *it clearly follows from (33) that (28) holds.*

Part II: Since $P_0 > 0$ *it directly follows from (33) that*

the ℓ_2 gain from the disturbance w_k to the performance output y_k is less than $\bar{\gamma}$.
Part III: Satisfaction of the LMIs (24d-24i) imply according to Lemma 1 satisfaction of the parameter-dependent matrix inqualities (11b) and (11c). Therefore, the properties of Theorem 1 hold at each sampling instant k, which implies satisfaction of the constraints (2) for all k.

In the next section we show via a simple simulation example that the online recalculation in the moving horizon fashion as presented in Theorem 2 is a suitable approach to tackle the given control problem.

4. SIMULATION EXAMPLE

To illustrate the effectiveness of the proposed ℓ_2 MHC scheme we consider the two-mass-spring model

$$x_{k+1} = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.1\dfrac{\mu}{m_1} & 0.1\dfrac{\mu}{m_1} & 1 & 0 \\ 0.1\dfrac{\mu}{m_2} & -0.1\dfrac{\mu}{m_2} & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ \dfrac{0.1}{m_1} \\ 0 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ \dfrac{0.1}{m_1} \\ 0 \\ 0 \end{bmatrix} w_k, \quad (34a)$$

$$y_k = [\,0\ 1\ 0\ 0\,]\, x_k, \quad (34b)$$

$$z_k = u_k, \quad (34c)$$

which is similar to the one presented in Yu et al. [2009a], Cuzzola et al. [2002], Kothare et al. [1996]. The model is obtained from a continuous time model using a first order Euler approximation with sampling time $\delta = 0.1s$. In (34) m_1 and m_2 are the masses and μ is the spring constant. The positions of the masses are represented by $x_{1,k}$ and $x_{2,k}$, whereas $x_{3,k}$ and $x_{4,k}$ describe their velocities. From the disturbance input w_k we know that it is energy bounded according to (3) with $\beta = 20$. For the simulation the constant masses $m_1 = 1$ and $m_2 = 1$ have been chosen. The spring constant has been assumed to be a time-varying function of the sampling instant k

$$\mu_k = 5 + \sin(0.5k). \quad (35)$$

Thus, for the uncertainty we have $\mu_k \in [4, 6]$. Introducing the parameters $\theta_{1,k} = 1 - \frac{\mu_k - 4}{2}$ and $\theta_{2,k} = \frac{\mu_k - 4}{2}$ system (34) can be written in the form as considered in this paper, i.e. the parameters $\theta_{i,k}$, $i = 1, 2$, satisfy condition (4) and the matrices A_i, $B = B_i$, and $G = G_i$, $i = 1, 2$, are as follows:

$$A_1 = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.4 & 0.4 & 1 & 0 \\ 0.4 & -0.4 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.6 & 0.6 & 1 & 0 \\ 0.6 & -0.6 & 0 & 1 \end{bmatrix}, \; G = \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \end{bmatrix}.$$

Furthermore, we have constant matrices $C = [0\ 1\ 0\ 0]$, $D = 0$, and $H = 0$. The control task is to design a moving horizon ℓ_2 controller as derived in Section 3 which minimizes the ℓ_2 gain from the disturbance input w_k to the performance output y_k. The controller has to satisfy input constraints of the form $|u_k| \leq 10$. For simplicity we do not consider state constraints in this paper, i.e. the matrices $E = [0\ 0\ 0\ 0]$ and $F = 1$ have to be chosen. Furthermore, we have $z_{max} = 10$. For the simulation we have chosen $\alpha = 250$. Figure 1 shows the obtained simulation results for the four states x_k, the control input u_k, and the disturbance w_k. The gray line represents the performance of a controller obtained after step 1 in Algorithm 1. The feedback matrices are applied to the system for all times and are not recalculated online. This controller guarantees the properties established by Theorem 1. The black line shows the results obtained by the moving horizon controller according to Algorithm 1, where at each time instant an online solution of the considered optimization problem is carried out. It is clearly visible that the disturbance is rejected much more efficiently due to the online recalculation of the control law. Although the constant feedback law which does not require online computations guarantees some useful properties, see Theorem 1, controller performance can be significantly improved when the control law is recalculated online since the controller can explicitly react on the disturbance. Furthermore, the current system state is taken into account in the optimization problem, which reduces conservativeness of the approach and allows for more aggressive control inputs. In many control problems this advantage overcomes the drawback of the online computational burden.

5. CONCLUSIONS

We presented a moving horizon ℓ_2 control approach based on the repeated online solution of a convex optimization problem subject to LMI conditions. At each time instant a parameter-dependent feedback law is calculated such that the ℓ_2 gain from the energy bounded disturbance to the performance output is minimized and state and input constraints are satisfied. A simulation example illustrated the effectiveness of the online recalculation of the control law leading to a significantly improved disturbance attenuation when compared to a static feedback law.

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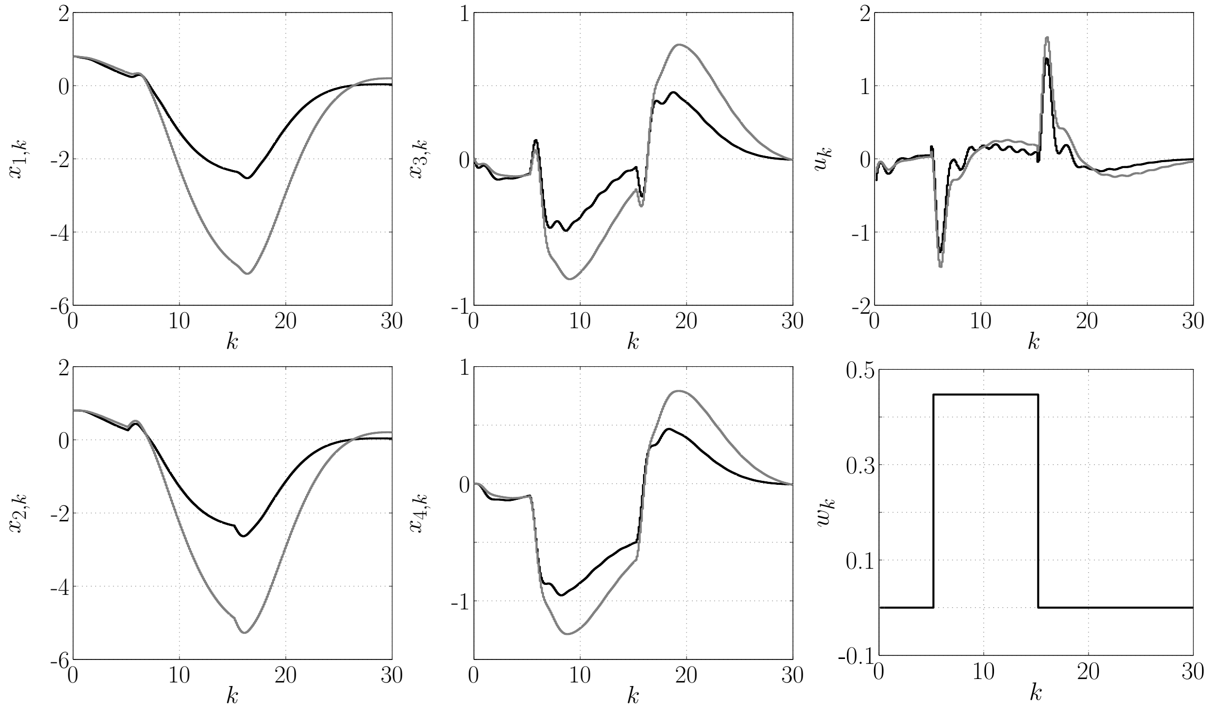

Fig. 1. Plots on the left: States x_k. Upper right plot: Control input u_k. Lower right plot: Disturbance input w_k. Black lines: Proposed moving horizon ℓ_2 controller. Gray lines: Static control law.

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